

arXiv: [math.PR/0000000](https://arxiv.org/abs/math.PR/0000000)

Local Pinsker inequalities via Stein's discrete density approach

Christophe Ley¹ and Yvik Swan

*Université Libre de Bruxelles
Département de Mathématique
Boulevard du Triomphe
Campus Plaine – CP210
B-1050 Brussels
e-mail: chrisley@ulb.ac.be*

*Université du Luxembourg
Faculté des Sciences, de la Technologie et de la Communication
Unité de Recherche en Mathématiques
6, rue Richard Coudenhove-Kalergi
L-1359 Luxembourg
e-mail: yvik.swan@uni.lu*

Abstract: Pinsker's inequality states that the relative entropy $d_{\text{KL}}(X, Y)$ between two random variables X and Y dominates the square of the total variation distance $d_{\text{TV}}(X, Y)$ between X and Y . In this paper we introduce generalized Fisher information distances $\mathcal{J}(X, Y)$ between discrete distributions X and Y and prove that these also dominate the square of the total variation distance. To this end we introduce a general discrete Stein operator for which we prove a useful covariance identity. We illustrate our approach with several examples. Whenever competitor inequalities are available in the literature, the constants in ours are at least as good, and, in several cases, better.

AMS 2000 subject classifications: Primary 60K35; secondary 94A17.

Keywords and phrases: Discrete density approach, Poisson approximation, scaled Fisher information, Stein characterizations, Total variation distance.

1. Introduction

Let $X \sim p$ and $Y \sim q$ be two real-valued random variables. The *relative entropy* between X and Y (a.k.a. *Kullback-Leibler divergence*, see [23]) is defined as

$$d_{\text{KL}}(Y||X) = d_{\text{KL}}(q||p) = \mathbb{E}_q \left[\log \left(\frac{q(Y)}{p(Y)} \right) \right], \quad (1.1)$$

where $\mathbb{E}_q[h(Y)]$ stands for the expectation of h under q . Although not a *bona fide* probability distance (absence of symmetry, no triangular inequality), Gibbs' inequality (see, e.g., [11])

$$d_{\text{KL}}(Y||X) \geq 0 \text{ with equality if and only if } p = q$$

¹Supported by a Mandat de Chargé de Recherche from the Fonds National de la Recherche Scientifique, Communauté française de Belgique. Christophe Ley is also a member of ECARES.

entails that $d_{KL}(Y||X)$ does indeed quantify a particular form of discrepancy (in terms of the entropies) between the law of X and that of Y . Moreover, letting $d_{TV}(X, Y)$ stand for the *total variation distance* between p and q (a precise definition is given in Section 3), the *Pinsker's inequality* (see, e.g., [11, 15])

$$2d_{TV}(X, Y) \leq \sqrt{2d_{KL}(Y||X)} \quad (1.2)$$

implies that the relative entropy dominates the total variation distance and thus, also, a large class of classical probability distances (including the Wasserstein distance, see e.g. [15] for an overview of the interrelations between probability metrics).

Fix $X = N$ a standard Gaussian random variable and consider absolutely continuous random variables Y with differentiable density q and finite variance which we set to 1. Estimates on $d_{KL}(Y||N)$ are typically obtained through control of the *Fisher information distance* (FID) between the law of Y and the Gaussian, which is defined as

$$J_N(Y) = E_q \left[\left(\frac{q'(Y)}{q(Y)} + Y \right)^2 \right] = I(Y) - 1, \quad (1.3)$$

with $I(Y) = E_q \left[(q'(Y)/q(Y))^2 \right]$ the Fisher information of Y . The FID can be viewed as a “local” version of the relative entropy (see, e.g., [3, 7–9, 20]). Trivially positive, it satisfies

$$J_N(Y) = 0 \text{ if and only if } Y \stackrel{L}{=} N$$

so that $J_N(Y)$ indeed quantifies discrepancy (this time in terms of the Fisher informations) between q and the Gaussian distribution. Finally the FID dominates the total variation distance

$$d_{TV}(N, Y) \leq \sqrt{2J_N(Y)} \quad (1.4)$$

(see [20, 32]) so that (similarly as the relative entropy) proximity between the law of Y and the Gaussian in terms of the Fisher information distance implies proximity in terms of a wide variety of more classical probability distances.

Fix $X = Po(\lambda)$ a rate- λ Poisson random variable and consider discrete random variables Y with probability mass function q on the positive integers. There exist at least two “local” versions of (1.1) which have been put to use in the literature on Poisson convergence, namely the *discrete Fisher information*

$$\mathcal{J}(Po(\lambda), Y) := E_q \left[\left(\frac{\lambda q(Y-1)}{q(Y)} - Y \right)^2 \right] \quad (1.5)$$

introduced in [6] (itself a generalization of an information functional presented in [21]) and the *scaled Fisher information*

$$\mathcal{K}(Po(\lambda), Y) := \lambda E_q \left[\left(\frac{(Y+1)q(Y+1)}{\lambda q(Y)} - 1 \right)^2 \right] \quad (1.6)$$

introduced in [22]. Both (1.5) and (1.6) are trivially positive and

$$\mathcal{J}(Po(\lambda), Y) = \mathcal{K}(Po(\lambda), Y) = 0 \text{ if and only if } Y \sim Po(\lambda)$$

so that these pseudo-distances indeed quantify a specific form of discrepancy between the density q and the Poisson distribution. The scaled Fisher information $\mathcal{K}(Po(\lambda), Y)$ dominates the relative entropy $d_{\text{KL}}(Po(\lambda), Y)$ (see [22]) and thus, by Pinsker's inequality (1.2),

$$d_{\text{TV}}(Po(\lambda), Y) \leq \sqrt{2\mathcal{K}(Po(\lambda), Y)}. \quad (1.7)$$

Consequently, as above, proximity in terms of the functional $\mathcal{K}(Po(\lambda), Y)$ entails proximity in terms of a wide variety of more classical probability distances.

Inequalities (1.4) and (1.7) are local versions of inequality (1.2) with respect to a fixed target distribution X . Moreover the three functionals (1.3), (1.5) and (1.6) are of the form

$$\mathcal{J}(X, Y) = \mathbb{E}_q [(r(p, q)(Y))^2]$$

for $r(p, q)$ a mean-0 functional which we interpret as a score function. In view of the fact that Pinsker's inequality is valid irrespective of the laws of X and Y , it is natural to enquire whether there exists some universal score function $r(p, q)$ whose variance $\mathcal{J}(X, Y)$ provides an informative “information distance” between the laws of X and Y such that (i) $\mathcal{J}(X, Y) \geq 0$ with equality if and only if $X \stackrel{\mathcal{L}}{=} Y$, and (ii) $\mathcal{J}(X, Y)$ satisfies the local Pinsker's inequality

$$d_{\text{TV}}(X, Y) \leq \kappa \sqrt{\mathcal{J}(X, Y)} \quad (1.8)$$

for κ some constant whose value only depends on the properties of the target distribution p .

A partial answer to this question is already known in case p and q are continuously differentiable probability density functions. Indeed in [24] we introduce the *generalized Fisher information distance*

$$\mathcal{J}(X, Y) := \mathbb{E}_q \left[\left(\frac{p'(Y)}{p(Y)} - \frac{q'(Y)}{q(Y)} \right)^2 \right]$$

which is a generalization of (1.3) to arbitrary densities p and q (note how, if p is the standard Gaussian density, we have $p'(x)/p(x) = -x$ so that we recover $\mathcal{J}(N, Y) = J_N(Y)$ the FID). Under assumptions on the supports of p and q we prove that $\mathcal{J}(X, Y)$ satisfies (1.8) and, for p the Gaussian, recover the constant $\kappa_p = \sqrt{2}$, and thus inequality (1.4).

The approach developed in [24] is reserved to continuously differentiable distributions on the real line, and the purpose of the present paper is to cover the case of discrete distributions. Before delving into the specifics of the discrete case, we start with an intuitive overview of our approach.

1.1. Sketch of the approach

Fix $[a, b] = \{a, a+1, \dots, b\}$ a collection of consecutive integers and consider a random variable $X \sim p$ with p a discrete probability distribution on $[a, b]$. Let Δ^η be the classical forward ($\eta = 1$) or backward ($\eta = -1$) difference operator on \mathbb{Z} (see (2.2) for a precise definition) and define the operator \mathcal{T}_p^η via the duality relationship

$$\mathbb{E}_p [(\mathcal{T}_p^\eta f)(X)g(X)] = \mathbb{E}_p [f(X)\Delta^\eta g(X)] \quad (1.9)$$

which we require to hold for all bounded functions g on \mathbb{Z} and all f belonging to some class $\mathcal{F}(p)$ which satisfy the appropriate boundary conditions (see Definition 2.1). Setting $g(x) = 1$ in (1.9) we immediately deduce that

$$\mathbb{E}_p [(\mathcal{T}_p^\eta f)(X)] = 0 \quad (1.10)$$

for all $f \in \mathcal{F}(p)$; in Theorem 2.1 we prove that the converse also holds true, i.e. if $Y \sim q$ and $\mathbb{E}_q [(\mathcal{T}_p^\eta f)(Y)] = 0$ for all $f \in \mathcal{F}(p)$ then $p = q$.

Operator \mathcal{T}_p^η is a generalization of the so-called *Stein operators* from the literature on Stein's method [4, 5, 10, 25] and the resulting characterization (Theorem 2.1) is a generalization of the so-called *density approach* adapted to the discrete setting, see e.g. [16, 31]. In the Appendix A.2, we will discuss specific examples for various choices of p and show how our operators contain many of the Stein operators which arise through other (sometimes more complex) methods, see e.g. [18].

The connection between Stein's method and information theory is implicit in the works [6, 19, 32] and is central to [24, 26]. See also the works [27–30] for alternative general considerations on the connexions between the two topics in the discrete setting. In this work as well we make use of a variation of this method, as follows. Given $X \sim p$ and $Y \sim q$ two random variables and l some test function, consider the solution f_l^p of the difference equation (a.k.a. Stein equation)

$$(\mathcal{T}_p^\eta f_l^p)(x) = l(x) - \mathbb{E}_p [l(X)]. \quad (1.11)$$

Much is known, from the literature on Stein's method, on the properties of the function f_l^p for several choices of target p (see, e.g., [5]). Taking expectations (w.r.t. q) on both sides of (1.11) and using fact (1.10) we get

$$\begin{aligned} \mathbb{E}_q [l(Y)] - \mathbb{E}_p [l(X)] &= \mathbb{E}_q [(\mathcal{T}_p^\eta f_l^p)(Y)] \\ &= \mathbb{E}_q [(\mathcal{T}_p^\eta f_l^p)(Y) - (\mathcal{T}_q^\eta f_l^p)(Y)] \end{aligned} \quad (1.12)$$

under the assumption that $f_l^p \in \mathcal{F}(q)$. Furthermore, it is easy to prove (see (2.4)) that we have the decomposition

$$(\mathcal{T}_p^\eta f)(x) - (\mathcal{T}_q^\eta f)(x) = f(x)r^\eta(p, q)(x) + \epsilon(x) \quad (1.13)$$

where ϵ has q -mean 0 and $r^\eta(p, q)$ is some functional of the densities p and q (and not of f) which, as we shall see, turns out to be a score function. Plugging (1.13) into (1.12) we get

$$\mathbb{E}_q [l(Y)] - \mathbb{E}_p [l(X)] = \mathbb{E}_q [f_l^p(Y)r^\eta(p, q)(Y)]. \quad (1.14)$$

Now, many probability distances (total variation distance, Kolmogorov distance, Wasserstein distance,...) can be written under the form

$$d_{\mathcal{H}}(X, Y) = \sup_{l \in \mathcal{H}} |\mathbb{E}_q[l(Y)] - \mathbb{E}_p[l(X)]|$$

for \mathcal{H} some class of functions (see, e.g., [25, Appendix C]). Taking suprema on either side of (1.14) we obtain

$$d_{\mathcal{H}}(X, Y) = \sup_{l \in \mathcal{H}} \mathbb{E}_q [|f_l^p(Y) r^n(p, q)(Y)|]. \quad (1.15)$$

We will use (Section 3) equality (1.15) to derive generalized Fisher information distances (for arbitrary discrete distributions) which we will prove to satisfy the local Pinsker's inequality (1.8) with an explicit constant κ . In particular we will introduce (i) the discrete Fisher information distance

$$\mathcal{J}_{\text{gen}}(X, Y) = \mathbb{E}_q \left[\left(\frac{q(Y-1)}{q(Y)} - \frac{p(Y-1)}{p(Y)} \right)^2 \right]$$

(Section 3.1) which generalizes (1.5) and (ii) the scaled Fisher information distance

$$\mathcal{K}_{\text{gen}}(X, Y) = \mathbb{E}_q \left[\left(\frac{p(Y)q(Y+1)}{p(Y+1)q(Y)} - 1 \right)^2 \right]$$

(Section 3.2) which generalizes (1.6). These are not the only discrete information distances that can be obtained by our approach, although they are the most relevant in view of the current literature on the topic. We illustrate (Section 3.3) an alternative construction in a specific setting related to the recent reference [14], and show that here as well our inequalities are competitive.

1.2. Outline of the paper

We start, in Section 2, by rigorously defining all the concepts appearing in Section 1.1. We also provide explicit conditions under which the manipulations are permitted. In Section 3 we discuss the local Pinsker's inequalities obtainable from (1.15) and provide several examples; we also compare our bounds with those already available in the literature. Finally the Appendix contains details, proofs and examples from Section 2.

2. Stein's density approach for discrete distributions

Let \mathcal{G} be the collection of probability mass functions $p : \mathbb{Z} \rightarrow [0, 1]$ with support $S_p := \{x \in \mathbb{Z} : p(x) > 0\}$ a discrete interval $[a, b] := \{a, a+1, \dots, b\}$ for $a < b \in \mathbb{Z} \cup \{\pm\infty\}$. We will, in the sequel, abuse language by referring to

probability mass functions as (discrete) densities. Throughout we adopt the convention that sums running over empty sets equal 0, and that

$$\frac{1}{p(x)} = \begin{cases} \frac{1}{p(x)} & \text{if } x \in S_p \\ 0 & \text{otherwise.} \end{cases} \quad (2.1)$$

Note how, in particular, convention (2.1) implies that $p(x)/p(x) = \mathbb{I}_{S_p}(x)$, the indicator of the support S_p . We will write $\mathbb{E}_p[l(X)] = \sum_{x \in S_p} l(x)p(x)$ for $p \in \mathcal{G}$ and l a p -summable function. Furthermore we introduce the η -difference operator

$$\Delta^\eta h(x) = \frac{1}{\eta} (h(x + \eta) - h(x)) \quad (2.2)$$

for all functions h taking their values on \mathbb{Z} . (Operators of the form (2.2) are not the only choice of “discrete derivative operator”; see e.g. [17] for an alternative).

Definition 2.1. Let $p \in \mathcal{G}$ and $|\eta| = 1$. We define (i) the collection $\mathcal{F}^\eta(p)$ of functions $f : \mathbb{Z} \rightarrow \mathbb{R}$ such that $\sum_{j=a}^b \Delta^\eta(f(j)p(j)) = 0$, and (ii) the operator $\mathcal{T}_p^\eta : \mathcal{F}^\eta(p) \rightarrow \mathbb{Z}^* : f \mapsto \mathcal{T}_p^\eta f$ given by

$$\mathcal{T}_p^\eta f : \mathbb{Z} \rightarrow \mathbb{R} : x \mapsto \mathcal{T}_p^\eta f(x) := \frac{1}{p(x)} \Delta^\eta(f(x)p(x)). \quad (2.3)$$

We call $\mathcal{F}^\eta(p)$ the class of η -test functions associated with p , and \mathcal{T}_p^η the η -Stein operator associated with p .

The first condition in Definition 2.1 (control of the functions at the edges of the support) ensures that we have the integration by parts formula

$$\mathbb{E}_p[(\mathcal{T}_p^\eta f)(X)g(X)] = -\mathbb{E}_p[f(X)\Delta^\eta g(X)]$$

for all functions g for which the above makes sense.

In particular, the class $\mathcal{F}^\eta(p)$ is tailored to ensure that $\mathbb{E}_p[\mathcal{T}_p^\eta f(X)] = 0$ for all $f \in \mathcal{F}^\eta(p)$. The following result (whose proof is deferred to the Appendix) shows that the converse holds true as well.

Theorem 2.1 (Discrete density approach). Fix $|\eta| = 1$ and let X be a discrete random variable with density $p \in \mathcal{G}$. Let Y be another discrete random variable with density $q \in \mathcal{G}$. Then $\mathbb{E}_q[\mathcal{T}_p^\eta f(Y)] = 0$ for all $f \in \mathcal{F}^\eta(p)$ if, and only if, either $\mathbb{P}(Y \in S_p) = 0$ or $\mathbb{P}(Y \in S_p) > 0$ and $\mathbb{P}(Y \leq z | Y \in S_p) = \mathbb{P}(X \leq z)$ for all $z \in S_p$.

Theorem 2.1 is a general *Stein characterization*. Expounding, for $\eta = 1$, the forward difference in (2.3) we get the same expression as [16, Equation (8)]. Our density approach and theirs are not equivalent, as described in [16, Remark 2.1]. The differences between their assumptions and ours are due to the “difference of a product” structure of (2.3). Examples wherein we apply Theorem 2.1 to specific choices of p and further details are discussed in the Appendix.

Fix, for the sake of convenience, $S_p = [0, \dots, M]$ and $S_q = [0, \dots, N]$, for some integers $0 \leq N \leq M \leq \infty$. Note in particular that we hereby ensure the

crucial assumption $S_q \subseteq S_p$. Now suppose that $\mathcal{F}^\eta(p) \cap \mathcal{F}^\eta(q) \neq \emptyset$ and choose some f in this intersection. Then, for this f , we can write

$$\begin{aligned} \mathcal{T}_p^\eta f(x) &= \mathcal{T}_q^\eta f(x) + \mathcal{T}_p^\eta f(x) - \mathcal{T}_q^\eta f(x) \\ &= \mathcal{T}_q^\eta f(x) + \frac{1}{\eta} \left(\frac{\Delta^\eta(f(x)p(x))}{p(x)} - \frac{\Delta^\eta(f(x)q(x))}{q(x)} \right) \\ &= \mathcal{T}_q^\eta f(x) + f(x+\eta) \frac{1}{\eta} \left(\frac{p(x+\eta)}{p(x)} - \frac{q(x+\eta)}{q(x)} \right) - \frac{1}{\eta} f(x) \mathbb{I}_{[N+1, \dots, M]}(x), \end{aligned} \quad (2.4)$$

where the indicator function equals 0 if $M = N$. Next let $l : \mathbb{Z} \rightarrow \mathbb{R}$ be a function such that both $\mathbb{E}_p[l(X)]$ and $\mathbb{E}_q[l(Y)]$ exist and consider the solution $f_l^{p,\eta}$ of the difference (Stein) equation

$$\mathcal{T}_p^\eta f(x) = l(x) - \mathbb{E}_p[l(X)]. \quad (2.5)$$

As in the proof of Theorem 2.1 (see identities (A.1) and (A.2)) it is easy to show that the solutions to (2.5) are given by

$$f_l^{p,+} : \mathbb{Z} \rightarrow \mathbb{R} : x \mapsto \sum_{k=0}^{x-1} (l(k) - \mathbb{E}_p[l(X)]) \frac{p(k)}{p(x)} \quad (2.6)$$

for $\eta = 1$ (the forward difference operator) and

$$f_l^{p,-} : \mathbb{Z} \rightarrow \mathbb{R} : x \mapsto \sum_{k=0}^x (l(k) - \mathbb{E}_p[l(X)]) \frac{p(k)}{p(x)} \quad (2.7)$$

for $\eta = -1$ (the backward difference operator). Recall that empty sums are set to 0. The functions $f_l^{p,\eta}$ as defined above trivially belong to $\mathcal{F}^\eta(p)$.

To pursue we need the following assumption.

Assumption A : The distributions p and q are such that the solutions $f_l^{p,\eta}$ of the Stein equation (2.5) satisfy $f_l^{p,\eta} \in \mathcal{F}^\eta(p) \cap \mathcal{F}^\eta(q)$ for $|\eta| = 1$.

For any given target p it is easy to determine conditions on q and l for Assumption A to be satisfied. These conditions are not restrictive.

Under Assumption A we can take expectations with respect to q on either sides of (2.4) applied to a solution of (2.5). Since $S_q \cap [N+1, \dots, M] = \emptyset$ we have $\mathbb{I}_{[N+1, \dots, M]}(Y) = 0$. Also $\mathbb{E}_q[\mathcal{T}_q^\eta f_l^{p,\eta}(Y)] = 0$, through Theorem 2.1 since $f_l^{p,\eta} \in \mathcal{F}^\eta(q)$ by Assumption A. Hence

$$\mathbb{E}_q[l(Y)] - \mathbb{E}_p[l(X)] = \mathbb{E}_q[\mathcal{T}_q^\eta f_l^{p,\eta}(Y)] + \mathbb{E}_q[f_l^{p,\eta}(Y+\eta)r^\eta(p,q)(Y)],$$

with

$$r^\eta(p,q)(x) := \frac{1}{\eta} \left(\frac{p(x+\eta)}{p(x)} - \frac{q(x+\eta)}{q(x)} \right). \quad (2.8)$$

We have proved the following result.

Lemma 2.1. *Take $p, q \in \mathcal{G}$ with $S_q \subseteq S_p$ and $l : \mathbb{Z} \rightarrow \mathbb{R}$ a function such that $E_p[l(X)]$ and $E_q[l(Y)]$ exist. Suppose moreover that Assumption A holds. Then*

$$E_q[l(Y)] - E_p[l(X)] = E_q[f_l^{p,\eta}(Y + \eta)r^\eta(p, q)(Y)], \quad (2.9)$$

with $f_l^{p,\eta}$ as in (2.6) and (2.7) and $r^\eta(p, q)$ as in (2.8).

Following the terminology from [1, 2, 16] we call (2.9) a Stein (or Stein-type) identity. Similarly as its counterpart [24, Lemma 3.2] in the absolutely continuous setting, Lemma 2.1 provides the connection between our version of the discrete density approach from Theorem 2.1 and discrete information inequalities.

3. Local Pinsker inequalities

As already mentioned in the introduction, a wide variety of probability metrics can be written under the form

$$d_{\mathcal{H}}(X, Y) = \sup_{l \in \mathcal{H}} |E_q[l(Y)] - E_p[l(X)]| \quad (3.1)$$

for some class of functions \mathcal{H} . In particular the total variation distance

$$d_{\text{TV}}(X, Y) := \frac{1}{2} \sum_{x \in \mathbb{N}} |p(x) - q(x)| = \sup_{l \in \{h\}} |E_q[l(Y)] - E_p[l(X)]|,$$

where the supremum in the second equality is taken over a set containing one single function, namely

$$h(x) := \frac{1}{2} (\mathbb{I}_{[p(x) \leq q(x)]} - \mathbb{I}_{[p(x) \geq q(x)]}) = \mathbb{I}_{[p(x) \leq q(x)]} - \frac{1}{2}.$$

Other distances such as the Kolmogorov, the Wasserstein, the supremum-distance or the L^1 -distance can also be written under the form (3.1) – we refer the reader to [15] or to [25, Appendix C] for an overview.

In view of (3.1), it is natural to take suprema on either side of (2.9) to deduce that, whenever Assumption A is satisfied, we have

$$d_{\mathcal{H}}(X, Y) = \sup_{l \in \mathcal{H}} |E_q[f_l^{p,\eta}(Y + \eta)r^\eta(p, q)(Y)]|. \quad (3.2)$$

Equation (3.2) is a very powerful identity as it permits to identify *natural* discrete information distances which uniformly dominate all probability distances of the form (3.1) through an inequality in which only the constant is distance-dependent. These inequalities being valid for virtually any choice (p, q) , we contend that their scope is comparable with that of Pinsker's inequality (1.2), this time for local versions of the (discrete) Kullback-Leibler divergence (1.1).

3.1. Fisher information inequalities via the backward difference operator

Choose the backward difference operator obtained for $\eta = -1$. Identity (2.9) spells out as

$$\mathbb{E}_q[l(Y)] - \mathbb{E}_p[l(X)] = \mathbb{E}_q[f_l^{p,-}(Y-1)r^-(p,q)(Y)] \quad (3.3)$$

with $r^-(p,q)(x) = \frac{q(x-1)}{q(x)} - \frac{p(x-1)}{p(x)}$ and with $f_l^{p,-}$ as in (2.7). Taking suprema on either side of (3.3) and applying Cauchy-Schwarz we obtain the following.

Theorem 3.1. *Take $p, q \in \mathcal{G}$ with $S_q \subseteq S_p$ and such that $\mathcal{F}^-(p) \cap \mathcal{F}^-(q) \neq \emptyset$. Let $d_{\mathcal{H}}(X, Y)$ be defined as in (3.1) for some class of functions \mathcal{H} , and suppose that for all $l \in \mathcal{H}$ the function $f_l^{p,-}$ defined in (2.7) exists and satisfies $f_l^{p,-} \in \mathcal{F}^-(p) \cap \mathcal{F}^-(q)$. Then*

$$d_{\mathcal{H}}(X, Y) \leq \kappa_{\mathcal{H}}^{p,-} \sqrt{\mathcal{J}_{\text{gen}}(X, Y)},$$

where

$$\mathcal{J}_{\text{gen}}(X, Y) := \mathbb{E}_q \left[\left(\frac{q(Y-1)}{q(Y)} - \frac{p(Y-1)}{p(Y)} \right)^2 \right] \quad (3.4)$$

is the generalized discrete Fisher information distance between the densities p and q , and

$$\kappa_{\mathcal{H}}^{p,-} := \sup_{l \in \mathcal{H}} \sqrt{\mathbb{E}_q \left[\left(f_l^{p,-}(Y-1) \right)^2 \right]}.$$

As an application suppose that p and q share the same support. Then we can write

$$\frac{q(x-1)}{q(x)} - \frac{p(x-1)}{p(x)} = \frac{\Delta^- p(x)}{p(x)} - \frac{\Delta^- q(x)}{q(x)}$$

so that (3.4) becomes

$$\mathcal{J}_{\text{gen}}(X, Y) = \mathbb{E}_q \left[\left(\frac{\Delta^- p(Y)}{p(Y)} - \frac{\Delta^- q(Y)}{q(Y)} \right)^2 \right]. \quad (3.5)$$

The distance (3.5) extends the Fisher information distance (1.5) to the comparison of any pair of densities p, q . Taking, in particular, p a Poisson target we retrieve

$$\begin{aligned} \mathcal{J}_{\text{gen}}(Po(\lambda), Y) &= \mathbb{E}_q \left[\left(\left(1 - \frac{Y}{\lambda} \right) - \frac{\Delta^- q(Y)}{q(Y)} \right)^2 \right] \\ &= \frac{1}{\lambda^2} \mathbb{E}_q \left[\left(Y - \frac{\lambda q(Y-1)}{q(Y)} \right)^2 \right], \end{aligned}$$

which in turn can be expressed as $\frac{\sigma^2}{\lambda^2} - \frac{2}{\lambda} + I(Y)$ with

$$I(Y) = \mathbb{E}_q \left[\left(\frac{\Delta^- q(Y)}{q(Y)} \right)^2 \right]$$

the functional proposed in [21] and λ, σ^2 the mean and variance of q (see also [6, equation 3.1]). In the particular case of the Poisson distribution, the function $f_l^{p_\lambda, -}(x-1)/\lambda$ is none other than the usual solution of the standard equation (A.3) for which we know (see [13, Theorem 2.3]) the estimate

$$\left\| \frac{f_l^{p_\lambda, -}(x-1)}{\lambda} \right\|_\infty \leq \left(1 - \sqrt{\frac{2}{e\lambda}} \right) \left(\sup_{i \in \mathbb{N}} l(i) - \inf_{i \in \mathbb{N}} l(i) \right);$$

this is useful when l is bounded as is the case, e.g., for the total variation distance. Moreover, this boundedness of $f_l^{p_\lambda, -}$ also ensures that Assumption A is satisfied whatever q (with support \mathbb{N}) we use, hence Theorem 3.1 can be applied. Since we always have

$$\kappa_{\mathcal{H}}^{p, -} \leq \sup_{l \in \mathcal{H}} \|f_l^{p, -}\|_\infty,$$

we conclude from Theorem 3.1 the information inequality

$$d_{\text{TV}}(Po(\lambda), Y) \leq \left(1 - \sqrt{\frac{2}{e\lambda}} \right) \sqrt{\sigma^2 - 2\lambda + \lambda^2 I(Y)}.$$

Note that, for $q = p_\lambda$, $I(Y) = \frac{1}{\lambda}$ and $\sigma^2 = \lambda$ so that $\sigma^2 - 2\lambda + \lambda^2 I(Y) = 0$, as expected.

The information distance (3.4) bears the defaults of its originator (1.5) : if p and q do not share the same support then \mathcal{J}_{gen} is infinite. In particular in the Poisson case the quantity for q with bounded support then, for some $k > 0$, we have $q(k) > 0$ with $q(x+1) = 0$ so that $I(Y) = +\infty$ (see e.g. the discussion at the beginning of [22, Section III]). One way to avoid this pathology is through a change in the derivative (2.2), as follows.

3.2. Fisher information inequalities for the forward difference operator

Choose the forward difference operator, that is take (2.2) this time with $\eta = 1$. Then $r^+(p, q)(x) = \frac{p(x+1)}{p(x)} - \frac{q(x+1)}{q(x)}$ and $f_l^{p, +}$ is of the form (2.6). If the target distribution p has support \mathbb{N} then $p(x)/p(x+1)$ is finite for all $x \in \mathbb{N}$ and the factorization

$$f_l^{p, +}(x+1)r^+(p, q)(x) = \left\{ f_l^{p, +}(x+1) \frac{p(x+1)}{p(x)} \right\} \left(1 - \frac{q(x+1)p(x)}{q(x)p(x+1)} \right) \quad (3.6)$$

is well-defined for all x . We introduce the scaled score function

$$r_{\text{sca}}(p, q)(x) = 1 - \frac{q(x+1)p(x)}{q(x)p(x+1)} \quad (3.7)$$

and the analog of Theorem 3.1 is obtained by yet another simple application of the Cauchy-Schwarz inequality to this factorization.

Theorem 3.2. *Take $p, q \in \mathcal{G}$ with $S_q \subseteq S_p$ and such that $\mathcal{F}^+(p) \cap \mathcal{F}^+(q) \neq \emptyset$. Let $d_{\mathcal{H}}(X, Y)$ be defined as in (3.1) for some class of functions \mathcal{H} , and suppose that for all $l \in \mathcal{H}$ the function $f_l^{p,+}$, as defined in (2.6), exists and satisfies $f_l^{p,+} \in \mathcal{F}^+(p) \cap \mathcal{F}^+(q)$. Then*

$$d_{\mathcal{H}}(X, Y) \leq \kappa_{\mathcal{H}}^{p,+} \sqrt{\mathcal{K}_{\text{gen}}(X, Y)}, \quad (3.8)$$

where

$$\kappa_{\mathcal{H}}^{p,+} := \sup_{l \in \mathcal{H}} \sqrt{\mathbb{E}_q \left[\left(f_l^{p,+}(Y+1) \frac{p(Y+1)}{p(Y)} \right)^2 \right]}$$

and

$$\mathcal{K}_{\text{gen}}(X, Y) = \mathbb{E} \left[(r_{\text{sca}}(p, q)(Y))^2 \right] = \mathbb{E}_q \left[\left(\frac{p(Y)q(Y+1)}{p(Y+1)q(Y)} - 1 \right)^2 \right]$$

is the generalized scaled Fisher information between the densities p and q .

In the case $p = Po(\lambda)$ we have $p_{\lambda}(x+1)/p_{\lambda}(x) = \lambda/(x+1)$ so that (3.8) becomes

$$\begin{aligned} d_{\mathcal{H}}(Po(\lambda), Y) &\leq \sup_{l \in \mathcal{H}} \sqrt{\mathbb{E}_q \left[\left(f_l^{p_{\lambda},+}(Y+1) \frac{\lambda}{Y+1} \right)^2 \right]} \sqrt{\mathcal{K}_{\text{gen}}(Po(\lambda), Y)} \\ &= \sup_{l \in \mathcal{H}} \sqrt{\mathbb{E}_q \left[\left(f_l^{p_{\lambda},+}(Y+1) \frac{\sqrt{\lambda}}{Y+1} \right)^2 \right]} \sqrt{\mathcal{K}(Po(\lambda), Y)} \end{aligned}$$

with $\mathcal{K}(Po(\lambda), Y) = \lambda \mathcal{K}_{\text{gen}}(Po(\lambda), Y)$ the scaled Fisher information distance (1.6). Using a Poincaré inequality, [22] show that, for q a discrete distribution with mean λ ,

$$d_{\text{TV}}(Po(\lambda), Y) \leq \sqrt{2\mathcal{K}(Po(\lambda), Y)}. \quad (3.9)$$

Our Theorem 3.2 allows to improve on this result, through the inequality (see again [13, Theorem 2.3])

$$\left\| \frac{f_l^{p_{\lambda},+}(x+1)}{x+1} \right\|_{\infty} \leq \left(1 - \sqrt{\frac{2}{e\lambda}} \right) \left(\sup_{i \in \mathbb{N}} l(i) - \inf_{i \in \mathbb{N}} l(i) \right);$$

indeed, this inequality combined with Theorem 3.2 yields (under the appropriate and more general conditions than in [22])

$$d_{\text{TV}}(Po(\lambda), Y) \leq \sqrt{\lambda} \left(1 \wedge \sqrt{\frac{2}{e\lambda}} \right) \sqrt{\mathcal{K}(Po(\lambda), Y)}. \quad (3.10)$$

For $\lambda < 2/e$, we get $1 \wedge \sqrt{\frac{2}{e\lambda}} = 1$ and hence the constant in (3.10) is $\sqrt{\lambda} < \sqrt{2/e}$; in case $\lambda > 2/e$, this constant equals $\sqrt{2/e}$. In both cases our constants improve on those from (3.9). More generally one easily sees that, for instance, in all examples considered in [22] our constants are better.

3.3. Other inequalities

In certain cases it is better to work directly from the Stein identity (3.2) without applying the Cauchy-Schwarz inequality. We illustrate this in the specific case of approximation of the rank distribution of random matrices over finite fields, as studied recently in [14].

Let M_n be chosen uniformly from $\text{Mat}(n, \theta)$ the collection of all $n \times n$ matrices over the finite field \mathbb{F}_θ of size $\theta \geq 2$. Let $Q_\theta^n = n - \text{rank}(M_n)$ and let Q_θ be its limiting version as $n \rightarrow \infty$. Both the distribution of Q_θ^n ($q_{k,n}$, $k = 0, \dots, n$, say) and that of Q_θ (q_k , $k \geq 0$, say) are known – see [14, equations (1), (2)]. These distributions satisfy the recurrence relations

$$\frac{q_{k-1}}{q_k} = \frac{(\theta^k - 1)^2}{\theta}, \quad k \in \mathbb{N} \quad \text{and} \quad \frac{q_{k-1,n}}{q_{k,n}} = \frac{(\theta^k - 1)^2}{\theta(1 - \theta^{-n+k-1})}, \quad k \in [0, n].$$

Using (3.2) with forward difference Δ^+ and factorization (3.6), the corresponding score function (3.7) simplifies to (for $p = q_k$ and $q = q_{k,n}$)

$$r_{\text{sca}}(Q_\theta, Q_\theta^n)(x) = \theta^{-n+x}$$

so that

$$d_{\mathcal{H}}(Q_\theta, Q_\theta^n) = \sup_{l \in \mathcal{H}} \left| \mathbb{E} \left[f_l^{\theta,+}(Q_\theta^n + 1) \frac{\theta}{(\theta^{Q_\theta^n+1} - 1)^2} \theta^{-n+Q_\theta^n} \right] \right| \quad (3.11)$$

with $f_l^{\theta,+}$ the solution to the difference equation (2.5) given by (2.6). See Appendix A.2 where we outline the setup of Stein's method via our Theorem 2.1 applied to this choice of distribution.

Inequality (3.11) allows to recover the upper bound from [14, Theorem 1.1]. Indeed it is shown there [14, Lemma 3.3] that

$$\mathbb{E} [\theta^{Q_\theta^n}] = 2 - \frac{1}{\theta^n} \quad \text{and} \quad \left\| \frac{f_l^{\theta,+}(x+1)}{(\theta^{x+1} - 1)^2} \right\|_\infty \leq \frac{1}{\theta^2} + \frac{1}{\theta^3},$$

if l is an indicator function. Plugging these facts into (3.11) we get

$$\begin{aligned} d_{\text{TV}}(Q_\theta, Q_\theta^n) &\leq \left(\frac{1}{\theta^2} + \frac{1}{\theta^3}\right) \theta^{-n+1} \mathbb{E} \left[\theta^{Q_\theta^n} \right] \\ &= \left(\frac{1}{\theta^2} + \frac{1}{\theta^3}\right) \theta^{-n+1} \left(2 - \frac{1}{\theta^n}\right) \\ &\leq \frac{2(1 + 1/\theta)}{\theta^{n+1}} \leq \frac{3}{\theta^{n+1}} \end{aligned}$$

for all $\theta \geq 2$; this is the upper bound from [14, Theorem 1.1].

One can also, using Hölder's inequality in (3.11), obtain bounds on the total variation distance in terms of higher moments $\mathbb{E} [\theta^{kQ_\theta^n}]$, $k \geq 1$. Initial computations show that the resulting inequalities are of equivalent rate but with constants depending on θ and bigger than 3. It would be interesting to enquire whether better inequalities are obtainable by exploiting the flexibility in (3.2). This is outside of the scope of the present article.

Appendix A: Details from Section 2

A.1. Proof of Theorem 2.1

If $\mathbb{P}(Y \in S_p) = 0$, the equivalence holds trivially so that we can take $\mathbb{P}(Y \in S_p) > 0$. We first check sufficiency. The equality $\mathbb{P}(Y \leq z \mid Y \in S_p) = \mathbb{P}(X \leq z)$ for all $z \in S_p$ can be rewritten as $\mathbb{P}(Y = z) = \mathbb{P}(X = z)\mathbb{P}(Y \in S_p)$, hence as $q(z) = p(z)\mathbb{P}(Y \in S_p)$, for all $z \in S_p$. Bearing in mind that the operator $\mathcal{T}_p^\eta f(x) = 0$ for all $x \notin S_p$, the sufficiency is easily established through

$$\mathbb{E}_q[\mathcal{T}_p^\eta f(Y)] = \mathbb{P}(Y \in S_p) \sum_{x \in S_p} \Delta^\eta(f(x)p(x)) = 0,$$

the last equality following by definition of the class $\mathcal{F}^\eta(p)$. Next, to see the necessity, define, for $z \in \mathbb{Z}$, the functions $l_z(k) := (\mathbb{I}_{(-\infty, z] \cap \mathbb{Z}}(k) - \mathbb{P}(X \leq z))\mathbb{I}_{S_p}(k)$ for $k \in \mathbb{Z}$ and define

$$f_z^{p,+1} : \mathbb{Z} \rightarrow \mathbb{R} : x \mapsto \frac{1}{p(x)} \sum_{k=a}^{x-1} l_z(k)p(k) \quad (\text{A.1})$$

and

$$f_z^{p,-1} : \mathbb{Z} \rightarrow \mathbb{R} : x \mapsto \frac{1}{p(x)} \sum_{k=a}^x l_z(k)p(k). \quad (\text{A.2})$$

Clearly these functions satisfy $\Delta^\eta(f_z^{p,\eta}(x)p(x)) = l_z(x)p(x)$ so that, in particular, $f_z^{p,\eta} \in \mathcal{F}^\eta(p)$ and

$$\mathcal{T}_p^\eta f_z^{p,\eta}(x) = l_z(x)$$

for all $x \in S_p$. Consequently, for this choice of test function we obtain

$$\begin{aligned} \sum_{x \in S_p} \mathcal{T}_p^\eta f_z^{p,\eta}(x) q(x) &= \sum_{x \in S_p} l_z(x) q(x) \\ &= \mathbb{P}(Y \leq z \cap Y \in S_p) - \mathbb{P}(Y \in S_p) \mathbb{P}(X \leq z), \end{aligned}$$

which, in combination with the hypothesis $\mathbb{E}_q [\mathcal{T}_p^\eta f_z^{p,\eta}(Y)] = 0$, finally yields $\mathbb{P}(Y \leq z | Y \in S_p) = \mathbb{P}(X \leq z)$ for all $z \in S_p$, whence the claim.

A.2. Examples of Stein operators

Theorem 2.1 extends and unifies many corresponding results from the literature, as will be shown through the following examples.

Take $p(x) = p_\lambda(x)$ the density of a mean- λ Poisson random variable. Then the class $\mathcal{F}^+(p) =: \mathcal{F}^+(\lambda)$ is composed of all functions $f : \mathbb{Z} \rightarrow \mathbb{R}$ such that (i) $x \mapsto \Delta^+(f(x)p_\lambda(x))$ is summable over \mathbb{N} and (ii) $f(0)p_\lambda(0) = \lim_{x \rightarrow \infty} f(x)p_\lambda(x)$ (which in most cases equals 0). In particular, $\mathcal{F}^+(\lambda)$ contains the set of bounded functions f such that $f(0) = 0$ (this border requirement is necessary in order to belong to \mathcal{F}^+ , see Definition 2.1(i)), for which simple computations show that

$$\mathcal{T}_\lambda^+ f(x) = \left(\frac{\lambda}{x+1} f(x+1) - f(x) \right) \mathbb{I}_{\mathbb{N}}(x).$$

This operator coincides with that discussed in [16, page 6]. One could also consider only functions of the form $f(x) = x f_0(x)$ for f_0 such that $x \mapsto x f_0(x) \in \mathcal{F}^+(\lambda)$ in which case no restriction on $f_0(0)$ (other than that it be finite) is then necessary to ensure the required border behaviour. Plugging such functions into (2.3) and simplifying accordingly we obtain

$$\tilde{\mathcal{T}}_\lambda^+ f(x) := (\lambda f_0(x+1) - x f_0(x)) \mathbb{I}_{\mathbb{N}}(x), \quad (\text{A.3})$$

which is none other than the standard operator for the Poisson distribution. Most authors refer to (A.3) as *the* Stein operator for the Poisson distribution although there are, of course, many more operators for this distribution which can be obtained from (2.3). One can, for instance, change the parameterization of the class $\mathcal{F}(\lambda)$ through “pre-multiplication” of the form $f(x) = c(x)f_0(x)$. See [16] for more on this approach. Another way of constructing Stein operators is by making use of the backward difference, for which the class $\mathcal{F}^-(p) =: \mathcal{F}^-(\lambda)$ is composed of all functions $f : \mathbb{Z} \rightarrow \mathbb{R}$ such that (i) $x \mapsto \Delta^-(f(x)p_\lambda(x))$ is summable over \mathbb{N} and (ii) $\lim_{x \rightarrow \infty} f(x)p_\lambda(x) = 0$. Here no border condition is necessary because $p_\lambda(-1) = 0$. For such f the operator becomes, after simplification,

$$\mathcal{T}_\lambda^- f(x) = \left(f(x) - \frac{x}{\lambda} f(x-1) \right) \mathbb{I}_{\mathbb{N}}(x)$$

which is, up to a scaling and a shift, equivalent to the standard operator (A.3).

Next let p be the density of S_n , the number of white balls added to the Pólya-Eggenberger urn by time n , with initial state $\alpha \geq 1$ white and $\beta \geq 1$ black balls. We know, e.g. from [16], that

$$p(k) = \mathbb{P}(S_n = k) = \binom{n}{k} \frac{(\alpha)_k (\beta)_{n-k}}{(\alpha + \beta)_n}$$

for $k = 0, \dots, n$, with $(x)_0 = 1$ and otherwise $(x)_k = x(x+1) \cdots (x+k-1)$ the rising factorial. Writing out the classes $\mathcal{F}^\eta(p)$ and the operators (2.3) in all generality for these distributions is of little practical or theoretical interest; in particular the resulting objects are hard to manipulate (see the discussion in [16]). It is much more informative to directly restrict one's attention to specific subclasses. For instance it is easy to see that $\mathcal{F}^+(p) =: \mathcal{F}^+(\alpha, \beta)$ contains all functions of the form $f(x) = x f_0(x)$ with f_0 bounded and, for these f , the operator is of the form

$$\tilde{\mathcal{T}}_{(\alpha, \beta)}^+ f(x) = \left(\frac{(n-x)(\alpha+x)}{\beta+n-x-1} f_0(x+1) - x f_0(x) \right) \mathbb{I}_{[0, n]}(x).$$

Likewise $\mathcal{F}^-(p) =: \mathcal{F}^-(\alpha, \beta)$ contains all functions of the form $f(x) = (n-x) f_0(x)$ with f_0 bounded and, for these f , the operator is of the form

$$\tilde{\mathcal{T}}_{(\alpha, \beta)}^- f(x) = \left((n-x) f_0(x) - f_0(x-1) \frac{x}{\alpha+x-1} (\beta+n-x) \right) \mathbb{I}_{[0, n]}(x).$$

Of course many variations on the above are imaginable. For instance one could also choose to consider functions of the form $f(x) = x(\beta+n-x) f_0(x)$; plugging these into (2.3) yields the operator discussed in [16, equation 7].

Thirdly we consider p belonging to the Ord family of distributions, that is we suppose that there exist $s(x)$ and $\tau(x)$ such that

$$\frac{p(x+1)}{p(x)} = \frac{s(x) + \tau(x)}{s(x+1)},$$

with $s(a) = 0$ (if finite) and $s(x) > 0$ for $a < x \leq b$. For an explanation on these notations see [31, equations (11) and (12)]. Writing out the classes $\mathcal{F}^\eta(p)$ and the operators (2.3) in all generality is again of little practical or theoretical interest. Note however that $\mathcal{F}^+(p) =: \mathcal{F}^+(s, \tau)$ contains all functions $f : \mathbb{Z} \rightarrow \mathbb{R}$ which are of the form $f(x) = f_0(x) s(x)$ with f_0 some bounded function. For these f , the operator writes out

$$\tilde{\mathcal{T}}_{(s, \tau)}^+ f(x) = ((s(x) + \tau(x)) f_0(x+1) - s(x) f_0(x)) \mathbb{I}_{[a, b]}(x),$$

and we retrieve the operator presented in [31]. Similarly for the backward operator we see that $\mathcal{F}^-(p) =: \mathcal{F}^-(s, \tau)$ contains all functions $f : \mathbb{Z} \rightarrow \mathbb{R}$ such that (i) $x \mapsto f(x) p(x)$ is bounded over S_p and (ii) $\lim_{x \rightarrow b} f(x) p(x) = 0$. For these f , the operator writes out

$$\tilde{\mathcal{T}}_{(s, \tau)}^- f(x) = \left(f(x) - \frac{s(x)}{s(x-1) + \tau(x-1)} f(x-1) \right) \mathbb{I}_{[a, b]}(x).$$

There are, of course, many variations on the approaches presented above.

Consider next any distribution p on $[0, n]$ satisfying the recurrence

$$a(x)p(x-1) = b(x)p(x) \text{ for all } x \in \mathbb{Z} \quad (\text{A.4})$$

with $a(x)$ and $b(x)$ some functions such that $a(x) \neq 0$ for all $x \in [0, n]$ and $b(0) = 0$. Suppose furthermore that $a(n+1) = 0$ (if n is finite). Then $\mathcal{F}^+(p)$ contains all functions of the form $f(x) = b(x)f_0(x)$ with f_0 some bounded function. For these f , the operator writes out

$$\mathcal{T}_{(a,b)}^+ f(x) = (a(x+1)f_0(x+1) - f_0(x)b(x)) \mathbb{I}_{[0,n]}(x),$$

and we hereby recover [14, Lemma 2.1]. The specific distributions studied in Section 3.3 are obtained by taking

$$a_n(x) = \theta(1 - \theta^{-n+x-1}) \text{ and } b_n(x) = (\theta^x - 1)^2$$

(distribution of Q_θ^n) and

$$a(x) = \theta \text{ and } b(x) = (\theta^x - 1)^2,$$

(distribution of Q_θ).

Finally choose p with support $[0, N]$ for some $N > 0$ and represent it as a Gibbs measure, that is, write

$$p(x) = \frac{e^{V(x)} \omega^x}{x! \mathcal{Z}} \mathbb{I}_{[0,N]}(x)$$

with N some positive integer, $\omega > 0$ fixed, V a function mapping $[0, N]$ to \mathbb{R} and $V(k) = -\infty$ for $k > N$, and \mathcal{Z} the normalizing constant. This is always possible, although there is no unique choice of representation (see [12]). Then $\mathcal{F}^\eta(p) =: \mathcal{F}^\eta(V, \omega)$ is composed of all functions $f : \mathbb{Z} \rightarrow \mathbb{R}$ which satisfy the summability requirements and such that either $f(0)p(0) = 0$ (if $\eta = 1$) or $f(N)p(N) = 0$ (if $\eta = -1$). In particular, $\mathcal{F}^+(V, \omega)$ contains functions of the form $f(x) = xf_0(x)$ with f_0 bounded and, for these f , the operator is of the form

$$\tilde{\mathcal{T}}_{(V,\omega)}^+ f(x) = \left(e^{V(x+1)-V(x)} \omega f_0(x+1) - xf_0(x) \right) \mathbb{I}_{[0,N]}(x); \quad (\text{A.5})$$

this corresponds to the Stein operator presented in [12]. Likewise if $N < \infty$ then $\mathcal{F}^-(V, \omega)$ contains functions of the form $f(x) = (N-x)f_0(x)$ with f_0 bounded and, for these f , the operator is of the form

$$\tilde{\mathcal{T}}_{(V,\omega)}^- f(x) = \left(f_0(x)(N-x) - x(N-x+1) \frac{e^{V(x-1)-V(x)}}{\omega} f_0(x-1) \right) \mathbb{I}_{[0,N]}(x)$$

and, if $N = \infty$, then $f(x) = f_0(x)$ with f_0 bounded suffices and the operator is equivalent to (A.5). Again a number of other parameterizations of the class $\mathcal{F}^\eta(V, \omega)$ can be considered, each leading to an alternative form of operator.

Acknowledgments

We thank two anonymous referees and the Associate Editor for their pertinent remarks which have led to substantial improvement of the paper.

References

- [1] G. Afendras, N. Papadatos, and V. Papathanasiou. The discrete Mohr and Noll inequality with applications to variance bounds. *Sankhyā*, 69(2):162–189, 2007.
- [2] G. Afendras, N. Papadatos, and V. Papathanasiou. An extended Stein-type covariance identity for the Pearson family with applications to lower variance bounds. *Bernoulli*, 17(2):507–529, 2011.
- [3] K. Ball, F. Barthe, and A. Naor. Entropy jumps in the presence of a spectral gap. *Duke Math. J.*, 119(1):41–63, 2003.
- [4] A. D. Barbour and L. H. Y. Chen. *An introduction to Stein's method*, volume 4 of *Lect. Notes Ser. Inst. Math. Sci. Natl. Univ. Singap.* Singapore University Press, Singapore, 2005.
- [5] A. D. Barbour, L. Holst, and S. Janson. *Poisson approximation*, volume 2 of *Oxford Studies in Probability*. The Clarendon Press Oxford University Press, New York, 1992. Oxford Science Publications.
- [6] A. D. Barbour, O. Johnson, I. Kontoyiannis, and M. Madiman. Compound Poisson approximation via information functionals. *Electron. J. Probab.*, 15:1344–1368, 2010.
- [7] A. R. Barron. Entropy and the central limit theorem. *Ann. Probab.*, 14(1):336–342, 1986.
- [8] L. D. Brown. A proof of the central limit theorem motivated by the Cramér-Rao inequality. In *Statistics and probability: essays in honor of C. R. Rao*, pages 141–148. North-Holland, Amsterdam, 1982.
- [9] E. Carlen and A. Soffer. Entropy production by block variable summation and central limit theorems. *Commun. Math. Phys.*, 140(2):339–371, 1991.
- [10] L. H. Y. Chen, L. Goldstein, and Q.-M. Shao. *Normal approximation by Stein's method*. Probability and its Applications (New York). Springer, Heidelberg, 2011.
- [11] T. Cover and J. Thomas. *Elements of Information Theory*, volume Second Edition. Wiley & Sons, New York, 2006.
- [12] P. Eichelsbacher and G. Reinert. Stein's method for discrete gibbs measures. *Ann. Appl. Probab.*, 18:1588–1618, 2008.
- [13] T. Erhardsson. Stein's method for poisson and compound poisson approximation. In *An introduction to Stein's method*, 2005.
- [14] J. Fulman and L. Goldstein. Stein's method and the rank distribution of random matrices over finite fields. *arXiv preprint arXiv:1211.0504*, 2012.
- [15] A. L. Gibbs and F. E. Su. On choosing and bounding probability metrics. *International Statistical Review / Revue Internationale de Statistique*, 70(3):pp. 419–435, 2002.

- [16] L. Goldstein and G. Reinert. Stein's method and the beta distribution. Preprint, arxiv:1207.1460, 2012.
- [17] E. Hillion, O. Johnson, and Y. Yu. A natural derivative on $[0, n]$ and a binomial poincaré inequality. *arXiv preprint arXiv:1107.0127*, 2011.
- [18] S. Holmes. Stein's method for birth and death chains. In *Stein's method: expository lectures and applications*, volume 46 of *IMS Lecture Notes Monogr. Ser.*, pages 45–67. Inst. Math. Statist., Beachwood, OH, 2004.
- [19] O. Johnson. *Information theory and the central limit theorem*. Imperial College Press, London, 2004.
- [20] O. Johnson and A. Barron. Fisher information inequalities and the central limit theorem. *Probab. Theory Related Fields*, 129(3):391–409, 2004.
- [21] I. Johnstone and B. MacGibbon. Une mesure d'information caractérisant la loi de poisson. In *Séminaire de probabilités*, volume XXI, pages 563–573. Springer, 1987.
- [22] I. Kontoyiannis, P. Harremoës, and O. Johnson. Entropy and the law of small numbers. *IEEE Trans. Info. Theory*, 51:466–472, 2005.
- [23] S. Kullback. A lower bound for discrimination information in terms of variation. *IEEE Trans. Info. Theory*, 4, 1967.
- [24] C. Ley and Y. Swan. Stein's density approach and information inequalities. *Electron. Comm. Probab.*, 18(7):1–14, 2013.
- [25] I. Nourdin and G. Peccati. *Normal approximations with Malliavin calculus : from Stein's method to universality*. Cambridge Tracts in Mathematics. Cambridge University Press, 2012.
- [26] I. Nourdin, G. Peccati, and Y. Swan. Entropy and the fourth moment phenomenon. *arXiv preprint arXiv:1304.1255*, 2013.
- [27] I. Sason. Entropy bounds for discrete random variables via coupling. Preprint, arXiv:1209.5259, 2012.
- [28] I. Sason. An information-theoretic perspective of the poisson approximation via the chen-stein method. *arXiv preprint arXiv:1206.6811*, 2012.
- [29] I. Sason. On the entropy of sums of bernoulli random variables via the chen-stein method. In *Information Theory Workshop (ITW), 2012 IEEE*, pages 542–546. IEEE, 2012.
- [30] I. Sason. Improved lower bounds on the total variation distance and relative entropy for the poisson approximation. *arXiv preprint arXiv:1301.7504*, 2013.
- [31] W. Schoutens. Orthogonal polynomials in Stein's method. *J. Math. Anal. Appl.*, 253(2):515–531, 2001.
- [32] R. Shimizu. On Fisher's amount of information for location family. In *A Modern Course on Statistical Distributions in Scientific Work*, pages 305–312. Springer, 1975.